Relaxed LMI Based designs for Takagi Sugeno Fuzzy Regulators and Observers
Poly-Quadratic Lyapunov Function approach

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Abstract—This paper addresses the analysis and design of the state and output feedback fuzzy control for the stabilization of the closed loop continuous time Takagi – Sugeno fuzzy system. The approach utilized in this paper is the so called poly Lyapunov function or fuzzy Lyapunov function. An ordinary parallel distributed compensation design technique is used for the state feedback and the output feedback stabilization control problems. Sufficient conditions for both cases in terms of linear matrix inequality are derived. The proposed design procedure subjected to fuzzy Lyapunov function cures the limitation of the previous results in dealing with the time derivative of premise membership functions and use the natural PDC control. The derived stability condition is more relaxed than the previous results. Moreover, the regulator and observer based stabilization problems are cast as a linear matrix inequality feasibility problem instead of the conventional bilinear matrix inequality. This problem is amenable to solve in the framework of the convex optimization. Finally, a simulation example demonstrates the advantage of the given technique.

Keywords—T-S fuzzy control, Lyapunov function, LMI

I. INTRODUCTION

Drawing on the development on the stability analysis and systematic design for the well known Takagi-Sugeno fuzzy control [1], which is a convenient and flexible tool for handling complex nonlinear systems where its consequent parts are linear systems connected by IF–THEN rules [2], interest in finding relaxed stability and stabilizability conditions for such a fuzzy control system has been increasing last decades.

In this case, state feedback control theory allows designing fuzzy controllers and fuzzy observers (3) by the way of the so-called parallel distributed compensation (PDC) [5]. Having the property of universal approximation, this approach includes the multiple model approach and can be seen also as Polytopic Linear Differential Inclusions (PLDI) [6].

The study of the stability of these models is done using a common Lyapunov function. Most of the works propose the use of a common quadratic Lyapunov function, i.e. \( V(x(t)) = x^T(t)Px(t) \) with \( P = P^T > 0 \) [7]. All these results can be stated as linear matrix inequalities (LMIs) that can be efficiently implemented and solved. The power of these results stems from the fact that the search for controller gains and Lyapunov function can be stated as a convex optimization problem in terms of linear matrix inequalities (LMI’s). Such optimization problems can be solved efficiently using publically available software [19]. Unfortunately, the standard LMI conditions for quadratic stability are often found to be conservative when applied to fuzzy systems [8]. Although this approach allowed us to apply convex optimization for solving the problems, it was mostly found to be conservative because of the “common” (or strict) structure of the Lyapunov function independent from the fuzzy weighting functions. Moreover, when a large number of subsystems are involved, common Lyapunov functions are inadequate to establish stability or synthesize controllers, by virtue of their conservativeness.

Recently, the relaxed stability and stabilizability conditions for fuzzy control systems were reported to release the conservativeness of the conventional conditions by considering the interactions among the fuzzy subsystems[8]-[9]. The relaxed conditions admitted more freedom in guaranteeing the stability of the fuzzy control systems and were found to be very valuable in designing the fuzzy controller, especially when the design problem involves not only stability, but also the other performance requirements such as the speed of response, constraints on control input and output, and so on [10].

Several approaches have been developed to overcome the above mentioned limitations. Piecewise quadratic Lyapunov functions were employed to enrich the set of possible Lyapunov functions used to prove stability[11],[12] and [15]. Multiple Lyapunov functions have been paid a lot of attention due to avoiding conservatism of stability and stabilizability [13],[14] and [16].

In [13], The stability of the continuous time open loop T-S fuzzy system is discussed using the so called multiple or fuzzy Lyapunov function. The problem of the time derivative of premise membership function for the fuzzy Lyapunov function that turn the stability condition for that system to not generally solved analytically or even numerically is solved by converting the upper bound constraint of the time derivative of the premise membership function into LMIs. The main limitation in this paper is that the initial states \( x(0) \) should be known in these new LMIs. Furthermore, for different initial states, we need to solve the LMI again, thus the stability conditions given in this paper are initial state dependent.

In [14] and [16], the stabilizability of the continuous time closed loop T-S fuzzy system is studied using the same
Takagi-Sugeno continuous fuzzy system [1]:
whose consequent parts are characterized by the following;
facilitate the paper, we assume that the reader is familiar with
supposed to be both controllable & observable respectively. To
time closed loop model based T-S fuzzy systems in both cases;
defining relaxed stabilitzability conditions for the continuous
bound of the time derivative of the premise membership
limitations appeared in [13][14] and [16] by designing an
present research.
output feedback analysis and design is hardly addressed in the
controller construction and its simplicity.
It is well known that the observer design is a very important
problem in control systems, however, in the context of fuzzy
control systems using fuzzy Lyapunov function, the fuzzy
output feedback design is hardly addressed in the
The primary motivation for this work is to avoid the
limitations appeared in [13][14] and [16] by designing an
ordinary PDC controller, which has already proven that it is
very simple and nature and does not contain any derivative
term that may be lead to instability, without using the initial
state dependent LMI conditions in dealing with the upper
bound of the time derivative of the premise membership
functions for both the fuzzy state feedback control and the
fuzzy observer based control. This paper is towards further
defining relaxed stabilitability conditions for the continuous
time closed loop model based T-S fuzzy systems in both cases;
the state feedback and the observer based control (output
feedback).

II. ORDINARY FUZZY STATE & OUTPUT FEEDBACK
CONTROL

A. Preliminaries
In the paper, the pairs \((A_i, B_i)\) & \((A_j, C_j)\) \(i \in \{1,...,r\}\) are
supposed to be both controllable & observable respectively. To
facilitate the paper, we assume that the reader is familiar with the
following; basic fuzzy setup consisting of \(r\) fuzzy rules
whose consequent parts are characterized by the following
Takagi-Sugeno continuous fuzzy system [1]:
\[
x(t) = \sum_{i=1}^{r} h_i(z(t)) (A_i x(t) + B_i u(t))
\] (1)
and for the nonlinear plant represented by (1), the fuzzy
controller is designed to share the same \(IF\) parts with the plant,
the overall nonlinear controller is represented as follows:
\[
u(t) = \sum_{i=1}^{r} h_i(z(t)) (-F_i x(t))
\] (2)
Moreover, the closed loop augmented T-S fuzzy Luenberger
observer FLO (The FLO is usually used for estimating the
system states \(x(t)\)) is represented as follows [17]:
\[
\dot{x} = \sum_{i=1}^{r} h_i(z(t))h_i(z(t))A_i x(t)
\] (3)
where
\[
A_{i} = \begin{bmatrix} A_i - B_i F_{i} & B_i F_{i} \\ 0 & A_i - L_i C_i \end{bmatrix}
and \(x(t) = [x(t) \ \ e(t)]\)
(4)
\(L_i\) is the observer gain for the \(i^{th}\) observer rule, \(F_{i}\) (for \(j = 1,2,...,r\)) is the controller gain for the \(j^{th}\) observer rule. \(e(t)\)
denotes the estimation error between \(x(t)\) & \(\hat{x}(t)\). \(z(t) = [z_1(t), z_2(t),...,z_r(t)]\) is the premise vector that may be states,
measureable external variables and/or time and \(h_i(z(t))\) is
the normalized weight for each rule, i.e. \(h_i(z(t)) \geq 0\) and
\[
\sum_{i=1}^{r} h_i(z(t)) = 1\]

B. Main Stability Condition
The main stability condition for the closed loop continuous
time fuzzy system (CLCTFS) is found in the literatures [9],
[20] as follows:

\textbf{Theorem 1:}
The fuzzy system (1) can be stabilized via the PDC controller
(2) if there exists a common positive definite matrix \(P\) such that

\[
P > 0
\] (5a)
\[G_{i} P + P G_{i}^{T} < 0 \quad (i = 1,...,r)\]
(5b)
\[G_{i} P + P G_{i}^{T} < 0 \quad (i = 1,...,r)\]
(5c)
\[
P^+ \geq 0 \quad (i = 1,...,r)\]
(6a)
\[
G_{i} P + P G_{i} (s-1) Q < 0 \quad (i = 1,...,r)\]
(6b)
\[
G_{i} P + P G_{i} < 0 \quad (i = 1,...,r)\]
(6c)
where \(s\) is the maximum of the number of the fuzzy subsystems
that are fired at an instant.
However, it is well known that in a lot of cases, a common
positive definite matrix \(P\) does not exist whereas the T-S model
(1) is stable. In this paper, the following fuzzy Lyapunov
function is employed, for the Takagi–Sugeno fuzzy system (1)
to relax the stabilization conditions (5) and (6):
\[
V(x(t)) = \sum_{i=1}^{r} h_i(z(t)) x^T(t) P_i x(t)
\] (7)
Where \(P_i\) is a positive–definite matrix. This candidate
Lyapunov function satisfies 1) \(V\) is \(C^2\), 2) \(V(0) = 0\) and \(V(x(t)) > 0\) for \(x(t) \neq 0\) and 3) \(\|x(t)\| \rightarrow \infty \Rightarrow V(x(t)) \rightarrow \infty\). The fuzzy
Lyapunov function shares the same membership functions with
the Takagi–Sugeno fuzzy model of a system. Applying fuzzy
Lyapunov function (7) on (1) instead of a single Lyapunov
function\(V(x) = x^T(t) P x(t)\) will result in the following theorem
Theorem 3 [14]: Assume that
\[ h_i(z(t)) < \phi_i \]
for \( \rho = 1, 2, ..., r \) - 1 where \( \phi_i > 0 \)
The fuzzy system (1) can be stabilized via PDC fuzzy controller (2) if there exist \( \varepsilon > 0, \phi_i, \) scalars \( s_1, s_2, ..., s_r, \) Positive definite matrices \( P_1, P_2, ..., P_r \) and matrices \( F_1, F_2, ..., F_r \) such that
\[ P_i \geq P_j, \quad i = 1, 2, ..., r - 1, \quad 0 \leq s_i, \quad i = 1, 2, ..., r \]
\[ \Omega_i = \begin{bmatrix} \frac{1}{3\varepsilon^2} (s_i + s_j) H_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{3\varepsilon^2} (s_i + s_j) H_{11} \end{bmatrix} \Omega_i > 0 \]
\[ \Omega_i = \begin{bmatrix} \frac{1}{3\varepsilon^2} (s_i + s_j) H_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{3\varepsilon^2} (s_i + s_j) H_{11} \end{bmatrix} \Omega_i > 0 \]

Assume that (8) as a feasibility problem offer a solution for the problem by converting the constraint (8) into solving the LMI again. Thus, the conditions in Theorem 4 are initial time derivatives of ))((

Proof: see [13] and [14].

Remark 1:
The authors in [13] and [14] comment on the difficulty for the selecting problem of \( \phi_i \) in practical control problems and give a solution for this problem by converting the constraint (8) into LMI. Hence, simultaneous solutions for theorem 3 and theorem 4 (given below) as a feasibility problem offer a solution for the stabilization of the fuzzy system (1).

Theorem 4 [14]
Assume that \( x(0) \) and \( z(0) \) are known. The constraint (8) holds if there exist positive definite matrices \( P_1, P_2, ..., P_r \), satisfying
\[ P_i x(t) = h_i(z(t)) F_i x(t) \]
\[ \phi_i, F_i > 0 \]
\[ \Omega_i > 0 \]
\[ \Omega_i = \begin{bmatrix} \frac{1}{3\varepsilon^2} (s_i + s_j) H_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{3\varepsilon^2} (s_i + s_j) H_{11} \end{bmatrix} \Omega_i > 0 \]

Proof: see [13] and [14].

Remark 2:
1) The initial states \( x(0) \) should be known in Theorem 4. In general, for different initial states, we need to solve the LMIs again. Thus, the conditions in Theorem 4 are initial states dependent. In the following section, for the design problem including some different initial states, the initial state independence condition can be deduced.

2) In [14] and [16] the problem of selecting of \( \phi_i \)'s is solved by proposing a new PDC design in the case where the time derivatives of \( h_i(z(t)) \) can be calculated from the states. The new PDC controller is of the following form
\[ u(t) = \sum_{i=1}^{r} h_i(z(t)) F_i x(t) \]
where \( T \) is an arbitrary PDC controller. In the following section, we use an ordinary PDC controller instead of the above one.

III. PDC STATE FEEDBACK CONTROL FOR CLCTFS

A. More Relaxed stability condition

Gurr & Vermeiren in [18] said a striking quotation: "It is necessary to change something either the control law or the Lyapunov function or both of them in order to relief the conservatism of the common quadratic Lyapunov function technique". In this paper we change the Lyapunov function to be a fuzzy Lyapunov function and keep the PDC control law in order to get more relaxed stability condition and very simple and straightforward controller synthesis.

Theorem 5: Given the closed loop system (1), if there exist \( \varepsilon > 0, \gamma > 0, \) scalars \( s_1, s_2, ..., s_r, \) and positive definite symmetric matrices \( P_1, P_2, ..., P_r \) such that
\[ P_i \geq S_j, \quad i = 1, 2, ..., r \]
\[ \Omega_i = \begin{bmatrix} \frac{1}{3\varepsilon^2} (s_i + s_j) H_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{3\varepsilon^2} (s_i + s_j) H_{11} \end{bmatrix} \Omega_i > 0 \]

Then the closed loop system (1) is quadratically stable. Proof: The ordinary state feedback PDC controller can be expressed as follows:
\[ u(t) = -\sum_{i=1}^{r} h_i(z(t)) \]

Substitute (13) into (1), \( \dot{V}(z(t)) \) can be obtained as follows:
\[ \dot{V}(z(t)) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t)) h_j(z(t)) \]
\[ \dot{V}(z(t)) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t)) h_j(z(t)) \]
\[ \dot{V}(z(t)) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t)) h_j(z(t)) \]
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\[ \dot{V}(z(t)) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t)) h_j(z(t)) \]

(14)
\[ V(t) = \sum_{\tau=1}^{\infty} \sum_{i,j=1}^{n} h_i(z_i) h_j(z_j) G_{ij} G_{ji} (P_i P_j + \frac{1}{6} \dot{P}_i \dot{P}_j) \]

Assume that
\[ h_i(z_i) = \sum_{m,n} w_{in}(z_i) \mu_{mn}, \]
where \( \sum_{m,n} w_{in}(z_i) \mu_{mn} = 1 \) and \( w_{in}(z_i) \geq 0 \). Then we have eq. (14).

\[ V(t) = \sum_{\tau=1}^{\infty} \sum_{i,j=1}^{n} h_i(z_i) h_j(z_j) G_{ij} G_{ji} (P_i P_j + \frac{1}{6} \dot{P}_i \dot{P}_j) \]

Note that, although the eq.(16) is complete the proof if it is negative definite, it is Bilinear Matrix Inequality (BMI) that is easily solved.

\[ (16) \]

The eq. (16) is negative if there exist \( \mu_{mn} \) from BMI into an LMI, which is used here. The idea is instead of dealing with the nonlinear condition in (16), we separate the matrix \( P \) into an additive term of \( \sum_{i,j} P_{ij} \), so that the condition changed to be linear condition in \( P \) (it transformed from BMI into an LMI) that is easily solved.

The eq. (16) is negative if there exist \( \varepsilon > 0 \) and \( \gamma > 0 \) such that,
\[ \mu_{mn}(P_i P_j) + \frac{1}{6(\gamma - 1)} \left( \dot{\mu}_i \dot{\mu}_j + \epsilon G_i G_j + \epsilon G_j G_i + \epsilon G_i G_j + \epsilon G_j G_i \right) < 0 \]

The left hand of (17) can be rewritten as:
\[ \mu_{mn}(P_i P_j) + \frac{1}{6(\gamma - 1)} \left( S_{i} + S_{j} + S_{ij} + S_{ij} + S_{ij} + S_{ij} \right) \]
where
\[ S_{i} = U_{i} P_{i} U_{i} = (G_{i} + \frac{1}{\gamma} P_{i}) (G_{i} + \frac{1}{\gamma} P_{i}) \]

Assume that \( \varepsilon \leq P \) & \( 1 \leq i \leq j \) then the following eq. holds:
\[ (16) \leq \mu_{mn}(P_i P_j) + \frac{1}{6(\gamma - 1)} \left( \frac{1}{3 \varepsilon} (r_i + r_j + r_{ij} + \frac{1}{\gamma} (r_i + r_j)) \right) \]

\[ V = 0 \] if (19) is negative definite matrix, (19) can be rewritten as:
\[ \frac{1}{3 \varepsilon (\gamma - 1)} (r_i + r_j + r_{ij} + \frac{1}{\gamma} (r_i + r_j)) \Omega_i - \mu_{mn}(P_i P_j) - \Omega_i = 0 \]

where
\[ \Omega_i = [\Omega_{i,0} \Omega_{i,1} \Omega_{i,2} \Omega_{i,3}] \]

The effectiveness of the suggested stability condition (12) compared with the basic well known stability conditions (5) and (6) can be illustrated by the following example.

B. Example 1: Consider the following fuzzy model:
\[ x(t) = \sum_{i=1}^{n} h_i(z_i(t))(A_i x(t) + B_i u(t)) \]
Where
\[ h_i(z_i(t)) = \frac{1 + \sin x_i(t)}{2} \]
\[ A_i = \begin{bmatrix} a & 1 \\ -1 & -2 \end{bmatrix} \]
\[ B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]
\[ y(t) = C x(t) \]
where \( C = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)

It is assumed that \( |x_i(t)| \leq \frac{\pi}{2} \) for \( i = 1, 2 \). A fuzzy controller is designed to have the local feedback gains with -J and -2 being the eigenvalues. The design parameters \( \mu_{mn} \) \( \{ \rho = 1, 2, ..., r - 1 \} \) can be calculated from (24, 25, 26 and 27) as follows: The time derivatives of membership functions is
\[ \dot{\mu}_i(x_i(t)) = \frac{\partial h_i(x_i(t))}{\partial x_i} \]

By substituting \( x_i(t) = [0, 0] \) into (23),

\[ \dot{h}_i(x_i(t)) = \frac{\partial h_i(x_i(t))}{\partial x_i} \]

Based on theorems 1, 2 and 5, the stability of the fuzzy control system is ensured for the parameter region shown in figures (1, 2 and 3).
From the membership function, the following values are obtained: $\mu_1 = 3.44$, $\mu_2 = -3.68$, $\mu_3 = 3.68$, $\mu_4 = -3.44$. These design parameters can be found using (26 and 27). (For more details see [16]). In the following, a PDC state feedback and PDC output feedback controllers are designed by solving the conditions in theorem 5 and 6.

B. PDC-State feedback Case

From conditions (20, 21 and 22) $P_1$ and $P_2$ can be obtained using MATLAB LMI toolbox [19].

$$P_1 = \begin{bmatrix} 1.1687 & -0.0056 \\ -0.0056 & 1.8540 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 1.1668 & -0.0203 \\ -0.0203 & 1.5640 \end{bmatrix}.$$  

Then the controller gains are found to be:

$$F_1 = [0.1209 \ 19.5362] \quad \text{and} \quad F_2 = [0.7485 \ 22.5395].$$  

Figure (4) shows the control result in case of state feedback controller.

C. PDC-Output feedback Case

From conditions (28, 29, 30 and 31) $P_{11}$ and $P_{22}$ can be obtained as follows:

$$P_{11} = \begin{bmatrix} 1.1704 & -0.0000 \\ -0.0000 & 1.2593 \end{bmatrix} \quad \text{and} \quad P_{22} = \begin{bmatrix} 1.2 & -0.0 \\ -0.0 & 0.4291 \end{bmatrix}.$$  

Then the controller and the observer gains are found to be:

$$F_1 = [-0.1052 \ 722.3681] \quad \text{and} \quad F_2 = [0.6334 \ 792.3601].$$  

$$I_1 = I_2 = \begin{bmatrix} -0.0000 \\ 4.2917 \end{bmatrix} \times 10^3.$$  

Figures (5&6) show the control result in case of output feedback controller for both the original states ($x_1$ & $x_2$) and the estimated states ($\hat{x}_1$ & $\hat{x}_2$).

Remark 3:

1) The dimension of the LMI stability condition (12) using ordinary PDC is 7nx7n while the dimension of the LMI stability condition in [14] and [16] using non PDC controller is 11nx11n, so our condition is more simple.

2) The derived LMI stability condition (12) doesn't depend on the initial state, thus, the condition in theorem 5 are initial states independent which is not the case in the stability condition of theorem 4.

3) The derived stabilizability condition for the closed loop continuous time TS system in theorem 5 using the polyquadratic Lyapunov function is pure LMI while the stabilizability condition for the same system in [15] is BMI that utilized a first order perturbation approximation method to be linearized and the solution of that BMI depend on the choice of initial values guaranteeing the existence of the solution.

4) Example 1 shows that theorem 5 suggested in this paper is more relaxed than theorem 1 and 2.

5) In this paper, we use an ordinary PDC controller that has already proven to be very simple and nature controller. In [14] and [16]a non PDC controller is used that is depend on the time derivatives of $h_i(z(t))$ which cannot always be calculated from the states. The non PDC controller works only for the case where the time derivatives of $h_i(z(t))$ can be calculated from the states.
VI. CONCLUSION

This paper considers the stabilization problems for the closed loop continuous time Takagi–Sugeno fuzzy system using the so-called fuzzy Lyapunov function. The proposed design procedure in this paper cures the limitation of the previous results in dealing with the time derivative of premise membership functions and uses the natural PDC instead of the non PDC control for the fuzzy state and output feedback control. The stability conditions in this paper avoid the use of the initial states $x(t)$, initial values for the unknown matrices $P$ & $F$ and the calculation of the time derivative of $h_{j}(x(t))$ from the states. The derived stability condition is more relaxed than the previous results. Moreover, it enables render the stability condition for the suggested state feedback and output feedback fuzzy control system to be pure LMI that is efficiently solved using the convex optimization techniques instead of the BMI condition that is hardly solved.

REFERENCES